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Polyhedral truncations as eutactic transformations

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An eutactic star is a set of N vectors in $\mathbb{R}^n$ $(N > n)$ that are projections of N orthogonal vectors in $\mathbb{R}^N$. First introduced in the context of regular polytopes, eutactic stars are particularly useful in the field of quasicrystals where a method to generate quasiperiodic tilings is by projecting higher-dimensional lattices. Here are defined the concepts of eutactic transformations (as mappings that preserve eutacticity) and of vector radiations (vectors that stem from the vectors of an eutactic star), which are used to describe and parameterize polyhedral truncations. The polyhedral truncations preserve eutacticity, a result of relevance to the faceting and habit-forming characteristics of quasicrystals.

1. Introduction

The concept of eutactic star has been particularly useful in the field of quasicrystals, where there are basically two methods to generate quasiperiodic tilings: the cut-and-projection method that generates quasiperiodic structures by projecting higher-dimensional lattices (Duneau & Katz, 1985), and the dualization method that does not require higher-dimensional spaces (for a review, see Gähler & Stampfli, 1993). Also, particular transformations of eutactic stars have been used to describe polyhedral transformations (Gancedo *et al.*, 1988), transformations between quasiperiodic and periodic tilings (Torres *et al.*, 1989*a*,*b*), structural changes in biological systems (Torres *et al.*, 2002) and small particles (Aragón, 1994). A generalization of these particular transformations has lead us to the concept of eutactic transformation. Here we formalize this concept and study a special transformation, called vector radiation, which we use to describe and parameterize polyhedral truncations. Since grain shapes of quasicrystals exhibit complex polyhedral truncations, the question of the relationship between eutacticity and external shapes (habits) in quasicrystals arises and is outlined in this work.

A *star* in $\mathbb{R}^n$ is a set of N vectors $\{\mathbf{a}_1, \ldots, \mathbf{a}_N\}$ in $\mathbb{R}^n$, with $n < N$. The star is called *eutactic* if there are N orthogonal vectors $\{\mathbf{u}_1, \ldots, \mathbf{u}_N\}$ in $\mathbb{R}^N$ and an orthogonal projector $P : \mathbb{R}^N \to \mathbb{R}^n$ such that

$$P(\mathbf{u}_i) = \mathbf{a}_i, \quad i = 1, \ldots, N.$$

As is well known, the orthogonal projector P is an idempotent and Hermitian $N \times N$ matrix that preserves the decomposition of $\mathbb{R}^N$ into a subspace E (where the eutactic star lies) and its orthogonal complement $E^{\perp}$.

Eutactic stars (from the Greek *eu* = good and *taxy* = arrangement) were studied by Hadwiger in the context of regular polytopes (Hadwiger, 1940). In a previous work, we have provided a eutacticity criterion that allows us to determine whether a given quasiperiodic structure can be obtained by projecting from a hypercubic lattice or a non-cubic lattice must be used instead (Gómez *et al.*, 1991).

A well known condition under which a given star is eutactic is due to Hadwiger (1940):

Theorem 1. *(Hadwiger)*. A star $\{\mathbf{a}_1, \ldots, \mathbf{a}_N\}$ in $\mathbb{R}^n$ is eutactic if and only if there is a number λ such that for all $\mathbf{x} \in \mathbb{R}^n$

$$\sum_{i=1}^{N} (\mathbf{x} \cdot \mathbf{a}_i)\mathbf{a}_i = \lambda \mathbf{x}.$$

In the special case where $\lambda = 1$, the star is said to be a *normalized eutactic star*.

An alternative characterization of eutacticity was given by Seidel (1978). In what follows, we define the concept of eutactic transformation as a mapping of a eutactic star that preserves eutacticity, and a particular eutactic transformation is studied.

2. Eutactic transformations

An important application of eutactic stars results from the possibility of continuously transforming structures defined as linear integer combinations of the vectors of a vector star. If we are able to transform the vector star, the transformation of the complete structure becomes straightforward. Particular cases of this kind of transformation have been used in several fields as mentioned in §1. In what follows, we formalize and generalize these results.

Let us assume that the N vectors of a star in $\mathbb{R}^n$ are functions of a scalar θ. We can then define a *star transformation* in $\mathbb{R}^n$ as a map σ that assigns to each $\theta \in \mathbb{R}$ the star

$$\sigma(\theta) = \{\mathbf{a}_1(\theta), \mathbf{a}_2(\theta), \ldots, \mathbf{a}_N(\theta)\}.$$

If the star is eutactic for every value of θ, then the map $\sigma : \mathbb{R} \to \mathbb{R}^n$ is called an *eutactic transformation*.

3. Vector radiations

Definition 1. For $M > 2$, an *M-radiation* of a star $\{\mathbf{a}_1, \ldots, \mathbf{a}_N\}$ in $\mathbb{R}^3$ (where all the vectors have the same length) is the unfolding of each $\mathbf{a}_i$ into M vectors, with the rotational symmetry of a regular polygon, located on a cone with axis $\mathbf{a}_i$ and half apex angle θ (Fig. 1).

From this definition, it is clear that the M-radiation of a star of N vectors is another star with $N \times M$ vectors. The following theorem states that an M-radiation (viewed as a mapping that associates to every θ the set of $N \times M$ vectors) is an eutactic transformation.

Theorem 2. For a given θ, the M-radiation of an eutactic star is eutactic.

Proof. Let $\{\mathbf{a}_i\}_{i=1}^N$ be a eutactic star in $\mathbb{R}^3$ $(3 < N)$. Then, by Theorem 1, for any $\mathbf{x} \in \mathbb{R}^3$, we have that

$$\sum_{i=1}^{N} (\mathbf{x} \cdot \mathbf{a}_i)\mathbf{a}_i = \lambda \mathbf{x}.$$

Now consider an M-radiation of the $\{\mathbf{a}_i\}_{i=1}^N$ into the set $\{\mathbf{b}_i^j\}_{i=1,j=1}^{N,M}$. We define vectors $\boldsymbol{\Delta}_i^j$ $(1 \le i \le N$ and $1 \le j \le M)$, shown in Fig. 1(b), by means of

$$\mathbf{b}_i^j = \gamma \mathbf{a}_i + \boldsymbol{\Delta}_i^j \qquad (1)$$

(for some fixed real number γ), where $\boldsymbol{\Delta}_i^j \cdot \mathbf{a}_i = 0$ (this is possible since all the $\mathbf{b}_i^j$, for a fixed i, have the same projection onto $\mathbf{a}_i$). All the $\mathbf{a}_i$ and all the $\boldsymbol{\Delta}_i^j$ have the same norm, that is, $\|\mathbf{a}_i\| = \|\mathbf{a}_1\|$ and $\|\boldsymbol{\Delta}_i^j\| = \|\boldsymbol{\Delta}_1^1\|$ for all i and j.

Let us call W_i the subspace of $\mathbb{R}^3$ spanned by $\{\boldsymbol{\Delta}_i^j\}_{j=1}^M$ and $W_i^\perp$ its orthogonal complement. Clearly, $\dim(W_i) = 2$ and $\dim(W_i^\perp) = 1$. By taking into account that (for a fixed i) the vectors $\{\boldsymbol{\Delta}_i^j\}_{j=1}^M$ point to the vertices of a regular polygon in W_i, then

$$\sum_{j=1}^{M} \boldsymbol{\Delta}_i^j = 0. \qquad (2)$$

Using this result, we have that, for any $\mathbf{x} \in \mathbb{R}^3$,

$$\begin{aligned}
\sum_{j=1}^{M}\sum_{i=1}^{N}(\mathbf{x}\cdot\mathbf{b}_i^j)\mathbf{b}_i^j &= \sum_{j=1}^{M}\sum_{i=1}^{N}[\mathbf{x}\cdot(\gamma\mathbf{a}_i + \boldsymbol{\Delta}_i^j)](\gamma\mathbf{a}_i + \boldsymbol{\Delta}_i^j)\\
&= \sum_{j=1}^{M}\sum_{i=1}^{N}\gamma^2(\mathbf{x}\cdot\mathbf{a}_i)\mathbf{a}_i + \gamma\sum_{j=1}^{M}\sum_{i=1}^{N}(\mathbf{x}\cdot\mathbf{a}_i)\boldsymbol{\Delta}_i^j\\
&\quad + \gamma\sum_{j=1}^{M}\sum_{i=1}^{N}(\mathbf{x}\cdot\boldsymbol{\Delta}_i^j)\mathbf{a}_i + \sum_{j=1}^{M}\sum_{i=1}^{N}(\mathbf{x}\cdot\boldsymbol{\Delta}_i^j)\boldsymbol{\Delta}_i^j\\
&= \sum_{i=1}^{N} M\gamma^2(\mathbf{x}\cdot\mathbf{a}_i)\mathbf{a}_i + \sum_{j=1}^{M}\sum_{i=1}^{N}(\mathbf{x}\cdot\boldsymbol{\Delta}_i^j)\boldsymbol{\Delta}_i^j\\
&= M\gamma^2\lambda\mathbf{x} + \sum_{j=1}^{M}\sum_{i=1}^{N}(\mathbf{x}\cdot\boldsymbol{\Delta}_i^j)\boldsymbol{\Delta}_i^j.
\end{aligned}$$

It follows that the star $\{\mathbf{b}_i^j\}$ is eutactic if and only if the star $\{\boldsymbol{\Delta}_i^j\}_{j=1}^M$ is eutactic. To prove this, notice that, for a fixed i,

$$\sum_{j=1}^{M} \mathbf{x} \cdot \boldsymbol{\Delta}_i^j = \sum_{j=1}^{M} \mathbf{x}_i^\parallel \cdot \boldsymbol{\Delta}_i^j,$$

where $\mathbf{x} = \mathbf{x}_i^\parallel + \mathbf{x}_i^\perp$, $\mathbf{x}_i^\parallel \in W_i$ and $\mathbf{x}_i^\perp \in W_i^\perp$. More explicitly,

$$\mathbf{x}_i^\perp = \frac{\mathbf{x}\cdot\mathbf{a}_i}{\mathbf{a}_i\cdot\mathbf{a}_i}\mathbf{a}_i, \qquad \mathbf{x}_i^\parallel = \mathbf{x} - \mathbf{x}_i^\perp.$$

Then,

$$\sum_{j=1}^{M}\sum_{i=1}^{N}(\mathbf{x}\cdot\boldsymbol{\Delta}_i^j)\boldsymbol{\Delta}_i^j = \sum_{j=1}^{M}\sum_{i=1}^{N}(\mathbf{x}_i^\parallel\cdot\boldsymbol{\Delta}_i^j)\boldsymbol{\Delta}_i^j = \sum_{i=1}^{N}\lambda_i\mathbf{x}_i^\parallel$$

because $\{\boldsymbol{\Delta}_i^j\}_{j=1}^M$ (for a fixed i) is eutactic in W_i (this in turn is true because, in the plane, vectors pointing to the vertices of a regular polygon form an eutactic star). Finally,

$$\begin{aligned}
\sum_{j=1}^{M}\sum_{i=1}^{N}(\mathbf{x}\cdot\boldsymbol{\Delta}_i^j)\boldsymbol{\Delta}_i^j &= \sum_{i=1}^{N}\lambda_i\left(\mathbf{x} - \frac{\mathbf{x}\cdot\mathbf{a}_i}{\mathbf{a}_i\cdot\mathbf{a}_i}\mathbf{a}_i\right)\\
&= \left(\sum_{i=1}^{N}\lambda_i\right)\mathbf{x} - \sum_{i=1}^{N}\lambda_i\frac{\mathbf{x}\cdot\mathbf{a}_i}{\mathbf{a}_i\cdot\mathbf{a}_i}\mathbf{a}_i\\
&= N\mu\mathbf{x} - \mu\sum_{i=1}^{N}\frac{\mathbf{x}\cdot\mathbf{a}_i}{\mathbf{a}_i\cdot\mathbf{a}_i}\mathbf{a}_i\\
&= N\mu\mathbf{x} - \frac{\mu}{\|\mathbf{a}_1\|^2}\lambda\mathbf{x}\\
&= \left(N\mu - \frac{\mu}{\|\mathbf{a}_1\|^2}\lambda\right)\mathbf{x},
\end{aligned}$$

where all the λ_i are equal among themselves, because the disposition of the γ are the same around any $\mathbf{a}_i$ (call the common value μ). Then, $\{\boldsymbol{\Delta}_i^j\}_{j=1}^M$ is eutactic and this proves the theorem.

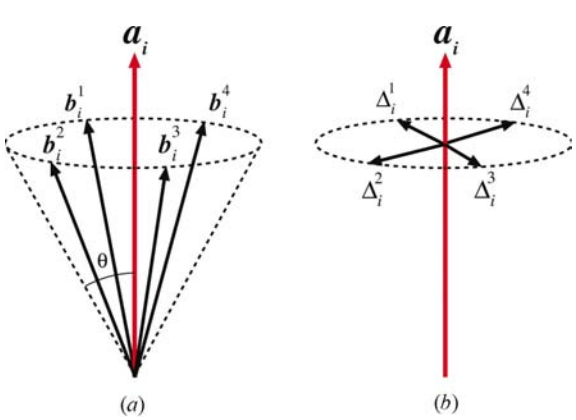

Figure 1
(a) Example of the 4-radiation of a vector $\mathbf{a}_i$; the vectors $\mathbf{b}_i^1, \ldots, \mathbf{b}_i^4$ point to the vertices of a square. (b) Cap of the cone showing the vectors $\boldsymbol{\Delta}_i^j$, defined in §3.

Definition 1 can be generalized *mutatis mutandis* to $\mathbb{R}^n$ and Theorem 2 is equally valid. The generalization considers a star $\{\mathbf{a}_1, \ldots, \mathbf{a}_N\}$ in $\mathbb{R}^n$ and the M-radiation of this n-dimensional star is the unfolding of each $\mathbf{a}_i$ into M vectors. In this case, we assume that the vectors $\boldsymbol{\Delta}_i^j$ $(1 \le j \le M)$, defined in (1), point to the vertices of a regular polytope that lies in a subspace of dimension $n-1$ orthogonal to $\mathbf{a}_i$. It guarantees the eutacticity of the star $\{\boldsymbol{\Delta}_i^j\}_{j=1}^M$ (see below) and the validity of (2).

4. The truncation of a polyhedron

The notion of eutacticity traces back to the Swiss mathematician L. Schläfli who (about 1858) called the vectors from the center to the vertices of any regular polytope eutactic stars. Hadwiger (1940) showed that eutactic stars are the orthogonal projection of *crosses*[1] into lower-dimensional spaces. In three-dimensional space, this implies that the vectors from the center to the vertices of regular polyhedra form eutactic stars. Additionally, Coxeter (1973) showed that the star associated with the rhombic dodecahedron, icosidodecahedron and triacontahedron are also eutactic.

An illustrative example of an M-radiation is the truncation of a regular polyhedron. The vectors pointing to the vertices of a truncated polyhedron can be viewed as the M-radiation of the vectors pointing to the vertices of the original polyhedron. This process is depicted in Fig. 2 with some examples; the truncation of a cube involves a 3-radiation and the truncation of an icosahedron is a 5-radiation. Notice that the truncation depth is controlled by the angle θ. A consequence of Theorem 2 is that, if vectors from the center to the vertices of a given polyhedron define a eutactic star (as in the case of regular polyhedra and the others mentioned above), the star of vectors from the center to the vertices of the resulting truncated polyhedron is eutactic. Consequently, in three-dimensional space, we have the result that *truncation preserves eutacticity*.

Complex truncations can also be described by means of M-radiations. In Fig. 3, we show three steps of the 10-radiation of an icosahedral vector star, producing a combination of vertex and edge truncations. For $\theta = 25.195^\circ$, this radiation generates a great rhombicosidodecahedron and, for $\theta = 33^\circ$, we have a truncated dodecahedron. Now, since two or more eutactic stars, with the same origin, form together a eutactic star (Coxeter, 1973), even more complex truncations can be achieved by combining two or more radiations, with the same or different half apex angle. The vector stars associated with the polyhedra obtained with this procedure are eutactic.

Physical implications of polyhedral truncation arise when studying faceting of crystals; densest atomic planes define the polyhedral external shape, or habit, of the crystal. This same idea is applied to quasicrystals, where the habits experimentally observed correspond to complex truncations of the polyhedron defined by the star of the associated quasiperiodic tiling. Theoretical studies of faceting in quasicrystals, based on the properties of a bond-oriented system with icosahedral symmetry (Ho *et al.*, 1987) or on the calculation of densest atomic planes (Aragón *et al.*, 1995) or on the theory of Wulff shapes (Böröczky *et al.*, 2000), have also predicted complex polyhedra with the symmetry of the associated quasiperiodic structure. One of the predicted shapes is shown in Fig. 3.

Some words concerning the possible relationship between faceting and eutacticity can be said here. Consider a particular face of a truncated polyhedron obtained by an M-radiation of the vector $\mathbf{a}_i$. This face lies in the plane W_i, defined in the proof of Theorem 2. Now $\{\boldsymbol{\Delta}_i^j\}_{j=1}^M$, the vectors from the center to the vertices of this face, point to the vertices of a regular polygon, *i.e.* they form a eutactic star in W_i. We then expect a direct relationship between the atomic packing in the facet defined by W_i and the eutactic star $\{\boldsymbol{\Delta}_i^j\}_{j=1}^M$. Voronoi's theorem on extreme lattices, which relates maximum of sphere packing density and eutacticity (Voronoi, 1909; Martinet, 2003), may play an important role.

5. Discussion

Particular vector radiations have been used to describe structural changes in different realms:

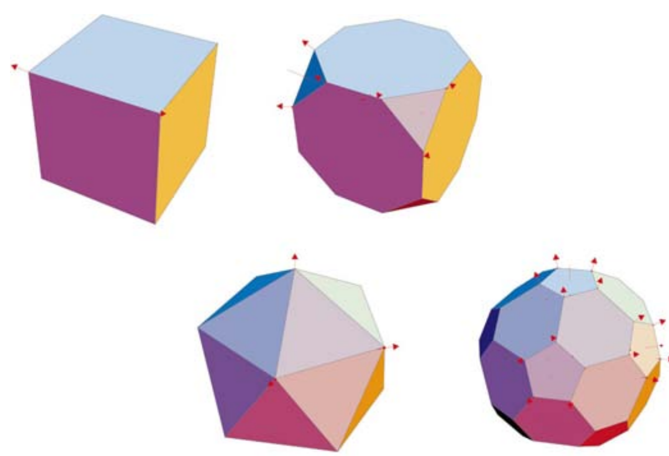

Figure 2
The truncation of a cube and of an icosahedron as examples of 3- and 5-radiations, respectively. In the first case, for example, the eight vectors from the center to the vertices of the cube form a eutactic star. When a 3-radiation is applied to this star, the truncated cube is obtained with θ controlling the truncation depth. The same reasoning is applied to the truncation of the icosahedron.

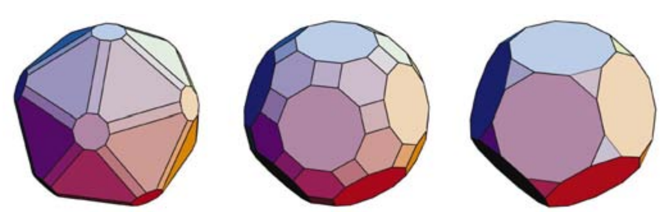

Figure 3
Three steps of the 10-radiation of an icosahedral star. The star composed by the 12 vertices of an icosahedron is eutactic and, when a 10-radiation is applied to this star, several truncations of the icosahedron are obtained depending of the value of the angle θ. From left to right, the values of θ are 10, 25.195 and 33°. Notice that the last two values produce a great rhombicosidodecahedron and a truncated dodecahedron, respectively.

[1] A *cross* in $\mathbb{R}^N$ is a set of N mutually perpendicular pairs of vectors $\pm\mathbf{e}_1, \ldots, \pm\mathbf{e}_n$ of equal length issuing from a fixed origin.

(i) transformations between quasiperiodic and periodic structures, ranging through all the observable quasiperiodic symmetries (Torres *et al.*, 1989*a*,*b*);

(ii) structural changes in small gold particles that, by irradiation of an electron beam, changed their shapes continuously from twinned icosahedral to cuboctahedral single crystals (Aragón, 1994);

(iii) changes in the five petaloid ambulacra of regular echinoids at different stages of their evolution (Torres *et al.*, 2002).

In this work, we have introduced the concept of eutactic transformation and showed that a general vector radiation, called M-radiation, defines an eutactic transformation. Additionally, we have provided a useful view of the truncation of a polyhedron as a vector radiation.

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